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CONVERGENCE CHARACTERISTICS OF FICTI-
TIOUS PLAY IN A SEARCH GAME

by

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Convergence Characteristics of Fictitious Play in a Search Game

by

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ABSTRACT

The convergence characteristics of an iterative method for solving area search games were investigated. This method, Fictitious Play, was first introduced by G. W. Brown and solves two-person zero-sum games by having each player sequentially select a pure strategy based on the combined past actions of his opponent. The Fictitious Play method was successfully implemented for an area search game in which two players, a searcher and a target, move independently through an area. In this game, the payoff is the number of detections of the target by the searcher. For each iteration of the game, an upper and lower bound on the value of the game were determined and as the number of iterations of the game increased, these bounds converged to the actual solution. In the games examined, the convergence of the bounds was closely approximated by a power function (αn^{β}), with large games converging more slowly. Because of the observed symmetrical convergence of the bounds, an accurate approximation of the value of the game was obtainable from the average of the upper and lower bounds.

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I. INTRODUCTION

A. BACKGROUND

The convergence properties of a computational method for solving finite matrix games was investigated. This method, Fictitious Play, was introduced by George W. Brown [Ref. 1] and is an iterative method based on the imagined play of the two game participants. At each fictitious play iteration, Brown's technique computes upper and lower bounds on the value of the game and approximates the optimal strategy for each player. For the games examined here, the convergence characteristics of the upper and lower bounds allow for an accurate approximation of the value of the game after relatively few iterations. Although the convergence rate of the bounds to the value of the game is slow, this iterative process allows the solving of large matrix problems which tend to become cumbersome with other common methods, e.g., Linear Programming. Very little is known about the convergence properties of Fictitious Play, and it is the purpose of this study to experimentally examine the rate of convergence for a specific two-person zero-sum area search game.

B. PROBLEM STATEMENT

This study was motivated by an area search game in which a searcher looks for an evading target, each moving among a finite number of cells in discrete time periods. The searcher and target each independently select a path through the search area (a pure strategy) or some probabilistic combination of paths (a mixed strategy). These paths are feasible combinations of cells serially connected over a time period T. That is, if the current cell is i, the next cell must be selected from a set C_i of neighboring cells. Although it will be assumed here, it is not necessary that cell i have the same set of neighbors for both searcher and target. For each game play, the searcher and target select feasible T-time period paths. The payoff is the expected number of times the searcher and target are in the same cell in the same time period. The searcher attempts to maximize and the target minimize this payoff.

Because the number of paths can be quite large, Fictitious Play was selected to solve this game. It soon became evident that the rate of convergence of Fictitious Play would determine whether or not it was a useful solution method.

C. PREVIOUS WORK

Fictitious Play was first introduced by G.W. Brown [Ref. 1] as an iterative process for solving finite two-person zero-sum games. Brown hypothesized that the rate of convergence to the value of the game was proportional to $1/n$, where n is the number of fictitious play iterations. Julia Robinson [Ref. 2] proved the process converged, thus formally demonstrating its potential validity as a solution method. J.M. Danskin [Ref. 3] showed that Fictitious Play applies to continuous two-person zero-sum games as well. S. Karlin [Ref. 4] hypothesized that the rate of convergence was $1/\sqrt{n}$, but further asserted that in practice it could be expected to converge more rapidly. No further work or relevant information on the convergence properties of Brown's Fictitious Play had been discovered up to the time of this study.

II. METHODOLOGY

A. FINITE MATRIX GAME

The area search game presented here can be represented as a finite matrix game. The elements of the payoff matrix are the number of detections of the target by the searcher, i.e., the number of time periods when the searcher and target are in the same cell. Figure 1 on page 4 depicts this matrix game, where

α_i = pure strategy i of searcher (a feasible T-time period path),

β_j = pure strategy j of target (a feasible T-time period path), and

r_{ij} = number of detections

for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,m$.

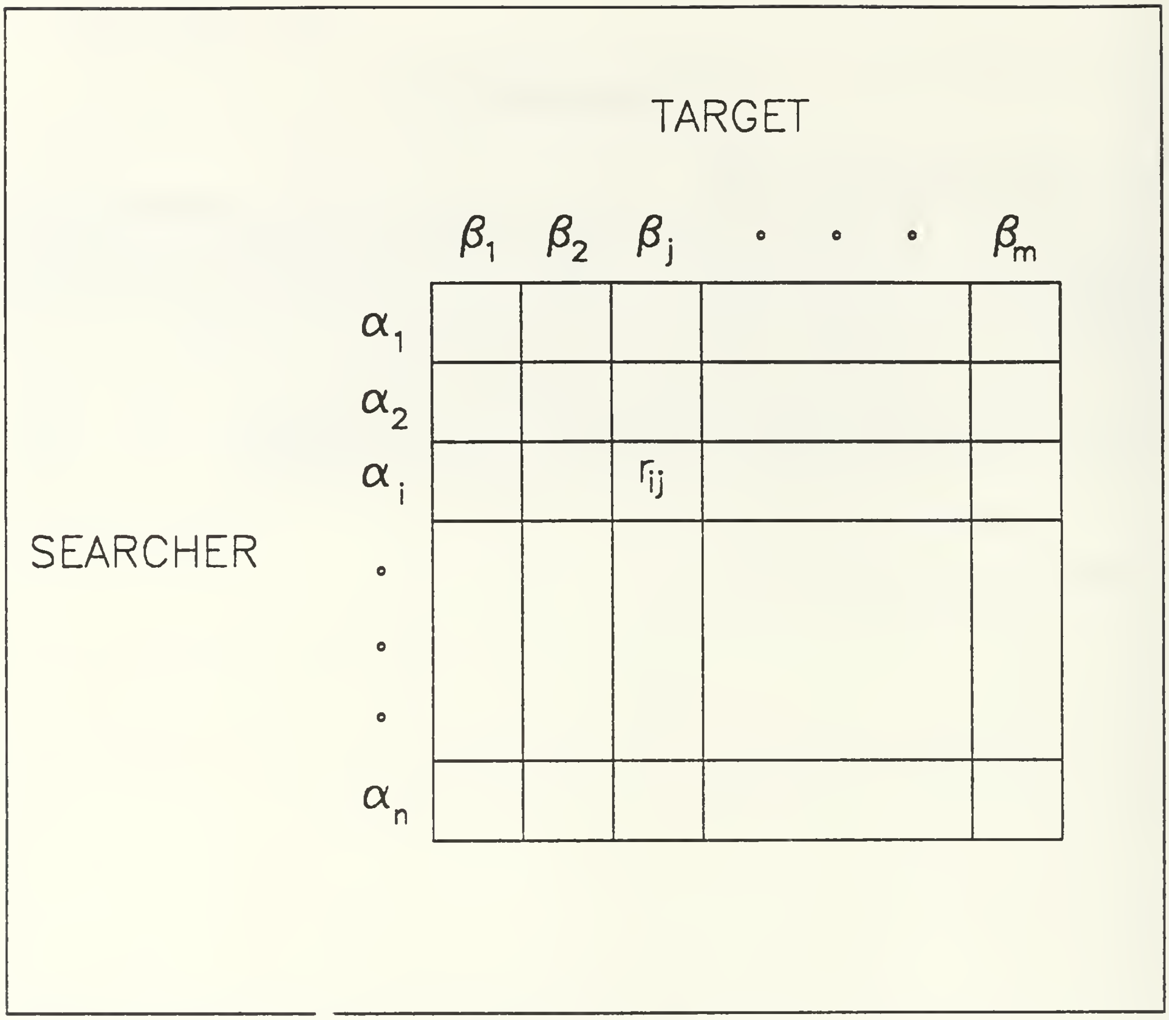

Figure 1. Finite Matrix Area Search Game

In this finite matrix game, the searcher can calculate a security level V_S , as

$$V_S = \max_i [\min_j r_{ij}].$$

Similarly the target can calculate a security level V_T , as

$$V_T = \min_j [\max_i r_{ij}].$$

In all cases, $V_S \leq V_T$ and if $V_S = V_T = V$, the game has an equilibrium or saddle point. Associated with this saddle point is the value of the game V, and optimal pure strategies α^* and β^* .

In games where equilibrium points do not exist, mixed strategies can be used to specify a value of the game. A mixed strategy is a set of pure strategies α_i , that are weighted with probabilities x_i , where

$$\sum_i x_i = 1 \quad \text{and} \quad x_i \geq 0.$$

For the searcher, a mixed strategy is denoted as $X = (x_1\alpha_1, x_2\alpha_2, \ldots, x_n\alpha_n)$, and for the target, it is denoted as $Y = (y_1\beta_1, y_2\beta_2, \ldots, y_m\beta_m)$. With mixed strategies, the security levels for the searcher and target are calculated respectively by:

$$V_S = \max_X \min_Y \sum_{i,j} x_i y_j r_{ij}$$

and

$$V_T = \min_Y \max_X \sum_{i,j} x_i y_j r_{ij}.$$

John von Neumann [Ref. 5] showed that

$$V_S = V_T = V\,,$$

where V is the value of the game. The searcher mixed strategy which achieves V is X^* , the searcher's optimal mixed strategy; and, the target mixed strategy which achieves V is Y^*, the target's optimal mixed strategy.

B. FICTITIOUS PLAY

Fictitious Play is an iterative method for approximating the V, X^* and Y^* for a two-person zero-sum game. This method was first introduced by George W. Brown [Ref. 1] and is conceptually described as follows:

> The iterative method in question can be loosely characterized by the fact that it rests on the traditional statistician's philosophy of basing future decisions on the relevant past history. Visualize two statisticians, perhaps ignorant of min-max theory, playing many plays of the same discrete zero-sum game. One might naturally expect a statistician to keep track of the opponent's past play and, in the absence of a more sophisticated calculation, perhaps to choose at each play the optimum pure strategy against the mixture represented by all the opponent's past plays. For calculation purposes the rule used here is that strategies will be named in turn for each side,

choosing at each turn a pure strategy which is optimal against the cumulative history of the opponent's play to date. [Ref. 1: p. 374]

The method is a relatively simple iterative process that directs a player to select an optimal pure strategy in response to the current empirical mixed strategy of his opponent. Applied to the area search game, an iteration of this procedure consists of the following steps.

1. Based on an equal weighting of all the searcher's pure strategies observed so far by the target, the target selects the best pure strategy response.
2. A lower bound on the value of the game is computed as the expected number of detections when the searcher plays his current mixed strategy and the target selects the best pure strategy response.
3. Based on an equal weighting of all the target's pure strategies observed so far by the searcher, the searcher selects the best pure strategy response.
4. An upper bound on the value of the game is computed as the expected number of detections when the target plays his current mixed strategy and the searcher selects the best pure strategy response. (Go to step 1 for next iteration)

The procedure begins with the target assuming an arbitrary searcher strategy. As the number of iterations of the game are increased, the upper and lower bounds on the value of the game converge toward the actual value, and any converging subsequence of the empirical mixed strategies is an optimal mixed strategy. The convergence rate has been observed to be quite slow, so an effective solution might require a large number of Fictitious Play iterations. The process is considered complete when the difference between the bounds on the value of the game is sufficiently small. At this point an approximate value of the game and approximate optimal strategies for both players are obtained.

The empirical mixed strategies after the kth iteration of Fictitious Play, A^k and B^k, are calculated from the relative frequencies of all the previously selected pure strategies of the searcher and target respectively. That is, consider the game that has been replicated k times and the searcher has selected pure strategies $(\alpha^1, \alpha^2, \ldots, \alpha^k)$, where α^j is the pure strategy chosen in the jth replication. If r_i denotes the number of times pure strategy α_i is used, then the pure strategy α_i is weighted with the relative frequency $\frac{r_i}{k}$. This results in the empirical mixed strategy

$$A^k = (\frac{r_1}{k}\alpha_1, \frac{r_2}{k}\alpha_2, \ldots, \frac{r_n}{k}\alpha_n)$$

for the searcher and similarly

$$B^k = \left(\frac{r_1}{k}\beta_1, \frac{r_2}{k}\beta_2, \ldots, \frac{r_m}{k}\beta_m\right)$$

for the target.

1. Convergence Rate

The convergence rate of the upper and lower bounds of the value of the game was first hypothesized by G.W. Brown [Ref. 1] to be $1/n$, where n is the number of iterations. He supported this hypothesis by relating the iterative method, as a difference equation, to a set of differential equations for which a convergence rate could be shown. The most recently found discussion in the literature on the convergence rate was presented by Samuel Karlin [Ref. 4]. He stated:

> It is conjectured that the process converges at a rate $1/\sqrt{k}$, where k is the number of iterations. In actual cases, it is found that the process is far more efficient than is expected theoretically.[Ref. 4: p. 183]

No reference was made to how or where this conjectured rate of convergence was determined. It appears that the rate of convergence of Fictitious Play is unclear and requires further investigation.

C. DYNAMIC PROGRAMMING

To use Fictitious Play to solve this area search game, both the searcher and target must be able to calculate the best pure strategy response to any mixed strategy of the opponent. If the area search game were small enough to allow a total enumeration of searcher and target pure strategies (i.e., the payoff matrix could be completely specified), then choosing the best pure strategy response would be simple. Assuming, for example, that the searcher is the row player, the best row i to play against a mixed strategy is

$$\underset{i}{\operatorname{argmax}} \left[\sum_j y_j r_{\cdot j} \right]_i$$

where

y_j = probability of target selecting column j, and

$r_{\cdot j}$ = column j of the payoff matrix.

The target could likewise find the best column response to any probabilistic combination of rows.

For the area search problem presented here, it is assumed that the large number of possible pure strategies (i.e., paths) available for the searcher and target makes total enumeration impractical. Another method must be used to determine pure strategy responses. The procedure employed is Dynamic Programming. Assume, for example, that the target plays a mixed strategy which is known to the searcher. That is, the searcher knows all the paths that the target might select and the probability of the target selecting each path. From this information the searcher computes

$$Y_t(i)\ ,\ i = 1,\ldots,N \ \text{ and } \ t = 1,\ldots,T\ ,$$

which is the probability of the target being in cell i in time period t. These probabilities contain all the information necessary for the searcher to select his best response. The searcher, as it turns out, does not care what mixed strategy the target uses as long as the searcher can determine the $Y_t(i)$ values.

The searcher can now use Dynamic Programming to compute the best pure strategy response to $Y_t(i)$. The recursion for $i = 1,\ldots,N$ and $t = 2,\ldots,T$ is

$$V_{t-1}(i) \;=\; \max_{j \in C_i}\{Y_t(j) \;+\; V_t(j)\} \ \text{ and}$$

$$d_{t-1}(i) \;=\; \operatorname*{argmax}_{j \in C_i}\{Y_t(j) \;+\; V_t(j)\},$$

where

$V_t(j)$ = maximum obtainable expected number of target detections from time t+1 to T when the searcher starts in time period t and is in cell j,

C_i = the set of cells accessible from cell i in one time period by the searcher, and

$d_t(i)$ = the best next cell to search given the searcher is in cell i in time period t.

The recursion begins with

$$V_T(i) = 0\ \ ,\ \ i = 1,\ldots,N.$$

By solving a similar dynamic program, the target can determine his best pure strategy response to any searcher cell occupancy probabilities, $X_t(i)$.

To demonstrate the validity of this recursion, it is observed that

$$
\begin{aligned}
V_{t-1}(i) = \max_{j \in C_i}\{ & E[\#\text{ detections at time } t \mid \text{search is in cell } j \text{ at time } t] \\
& + E[\#\text{ detections from time } t+1 \text{ to } T \mid \text{search is in cell } j \text{ at} \\
& \quad \text{time } t \text{ and conducted optimally from time } t+1 \text{ to time } T]\} \\
= & \max_{j \in C_i}\{Y_t(j) + V_t(j)\}.
\end{aligned}
$$

The first equality follows from the definition of $V_t(j)$ and the fact that the expected value of a sum of random variables is the sum of the expected values. The second equality results from conditioning on the target's cell at time t and (again) the definition of $V_t(j)$.

It is noted that the searcher's problem is that of finding the longest (i.e., most profitable) path through the N x T acyclic network in Figure 2 on page 10. When the searcher reaches node j in time period t, the payoff $Y_t(j)$ is received. Arcs connect each cell i with all cells in the set of accessible cells C_i.

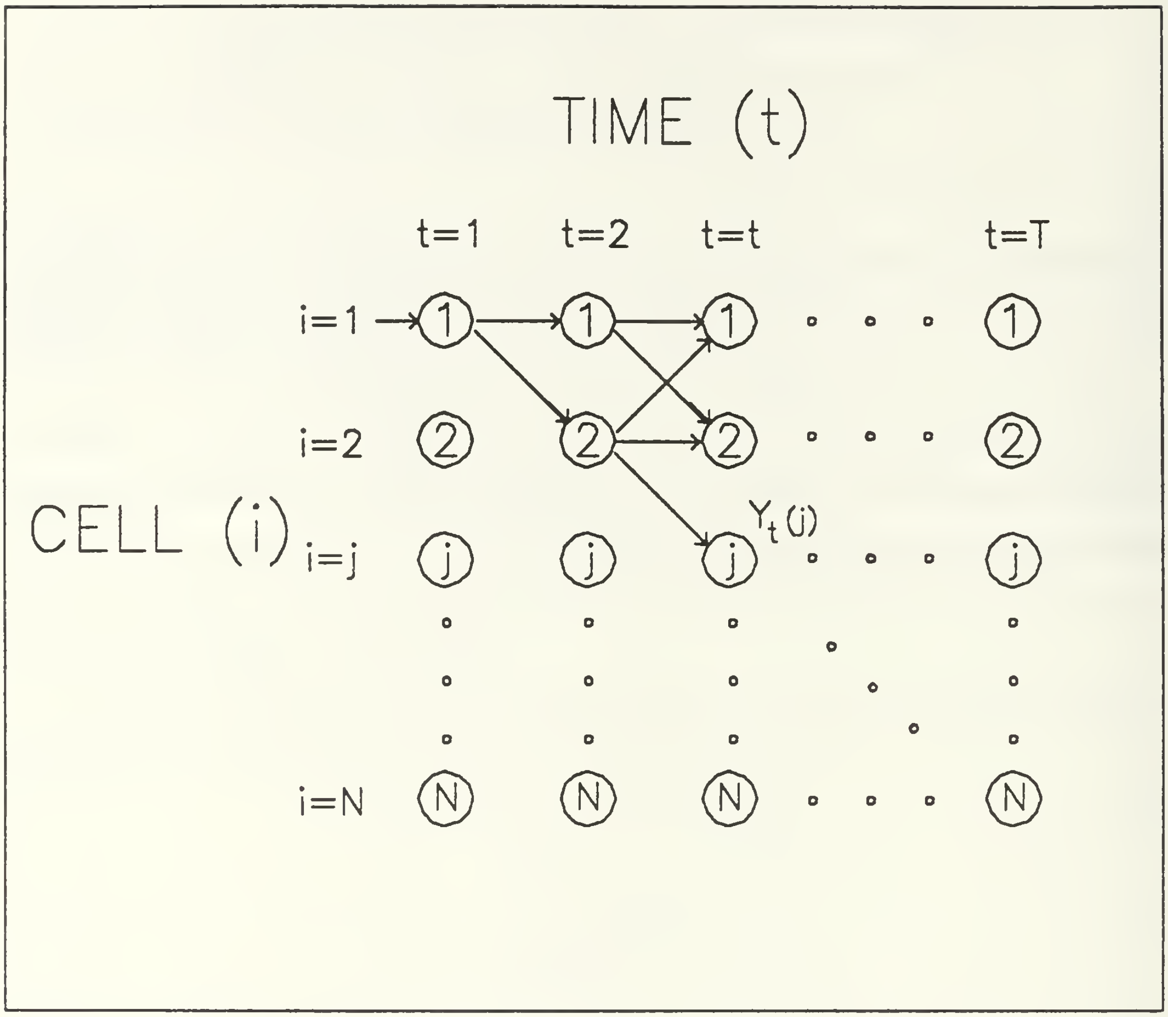

Figure 2. Network of Searcher's Path Through Search Area

1. Updating Cell Occupancy Probabilities, $X_t(i)$ and $Y_t(i)$

In each iteration of fictitious play, the searcher and target update $Y_t(i)$ and $X_t(i)$ respectively, based on the number of iterations performed so far and the opponents most recently observed pure strategy. Assigning equal weights at all observed pure strategies, the update procedure is straight forward. If the searcher was in cell i at time t in the most recently observed pure strategy, then

$$X_t^k(i) = \left(\frac{k-1}{k}\right) X_t^{k-1}(i) + \frac{1}{k},$$

and if the searcher was not in cell i at time t ,

$$X_t^k(i) = \left(\frac{k-1}{k} \right) X_t^{k-1}(i).$$

Likewise, if the target was in cell i at time t in the most recently observed pure strategy, then

$$Y_t^k(i) = \left(\frac{k-1}{k} \right) Y_t^{k-1}(i) + \frac{1}{k},$$

and if the target was not in cell i at time t,

$$Y_t^k(i) = \left(\frac{k-1}{k} \right) Y_t^{k-1}(i).$$

Here k is the current iteration number and $X_t^k(i)$ and $Y_t^k(i)$ are the empirical cell occupancy probabilities after the kth iteration.

III. DATA GENERATION

A. PROGRAM DESCRIPTION

The upper and lower bound data required for this study were generated by a modified version of an existing computer program provided by Professor J. Eagle at the Naval Postgraduate School in Monterey, California. The modified version was developed to allow a greater flexibility for variable manipulation. Conceptually the program was divided into three major areas dealing with Game Theory, Fictitious Play and Dynamic Programming.

1. Finite Matrix Game

The initial set up of the finite matrix game required inputs from the user that included search area size, duration of search, replications of game and initial strategy of players. Within this portion, all the possible decision paths (states) for adjacent cell path movements during a time period (stage) were determined. The initial inputs and adjacent cell paths were required for transition into the fictitious play portion.

2. Fictitious Play

The Fictitious Play portion was the driver of the iterative process. It was responsible for determining the mixed strategies of the players and evaluating the bounds on the value of the game. As the best pure strategies were determined, the bounds were reevaluated. The iterative process was started by the selection of an optimal pure strategy for the target against the searcher's initial inputed strategy. A new lower bound on the value of the game was computed. If it was greater (i.e., tighter) than the current lower bound, it was retained. Otherwise the new lower bound was ignored. A mixed strategy for the target was then determined from the old mixed strategy and the pure strategy just selected. The process of selecting the best pure strategy, calculating and evaluating the opponent's bound on the game, and determining a new mixed strategy was then accomplished for the searcher against the target's current mixed strategy. Conducting this process once for each player constituted a replication of the game. The determination of the optimal pure strategies and the bounds on the value of the game for each player required the Dynamic Programming portion of the program.

3. Dynamic Programming

Dynamic Programming provided the optimization procedure for the iterative process and was considered the optimizer portion. It determined the best search paths

for each player in response to the mixed strategies of his opponent. The payoff (expected number of detections) was maximized for the searcher and minimized for the target. These paths became the optimal pure strategies required for the calculation of empirical mixed strategies in the iterative process. Associated bounds on the value of the game were calculated for each iteration.

B. PROGRAM VALIDATION

The computer program was validated with results from a Linear Programming solution to the area search game, provided by Professor A. Washburn at the Naval Postgraduate School in Monterey, California. A comparison of the Linear Programming solutions and the Fictitious Play approximations is presented in Table 1. The fictitious play approximations were the computed midpoints between the upper and lower bounds on the value of the game after 50,000 replications. The difference between the solutions of the two approaches is represented as an absolute value. The validity of the fictitious play computer program was supported by these results.

Table 1. **COMPARISON OF LINEAR PROGRAMMING AND FICTITIOUS PLAY SOLUTIONS** (Value of the Game)

MATRIX SIZE	# TIME PERIODS	LINEAR PROGRAMMING	FICTITIOUS PLAY	ABSOLUTE DIFFERENCE
1x6	11	1.2900	1.2902	0.0002
3x3	6	0.3548	0.3550	0.0002
4x4	8	0.2553	0.2554	0.0001
5x5	10	0.2012	0.2015	0.0003
6x6	12	0.1666	0.1665	0.0001

IV. DATA ANALYSIS

A. CONVERGENCE PROPERTIES

In validating the Fictitious Play approach, it became apparent that an examination of convergence properties was needed. The only relevant information available pertained to convergence rates and was conflicting [Ref. 1,4]. In order to reasonably predict a solution, an understanding of convergence characteristics was required. The focus of the study became the investigation of convergence properties with specific emphasis on convergence symmetry and rate.

B. CONVERGENCE SYMMETRY

The upper and lower bounds on the value of the game converge to a solution as the number of replications increases. This was proven mathematically by J. Robinson [Ref. 2]. It was observed in this study that the bounds tended to converge symmetrically. Graphically this is displayed for a 4x4 matrix in Figure 3. This characteristic was present in all cases examined, which included various matrix sizes, shapes and initial player positionings. Additional graphic presentations are located in Appendix A.

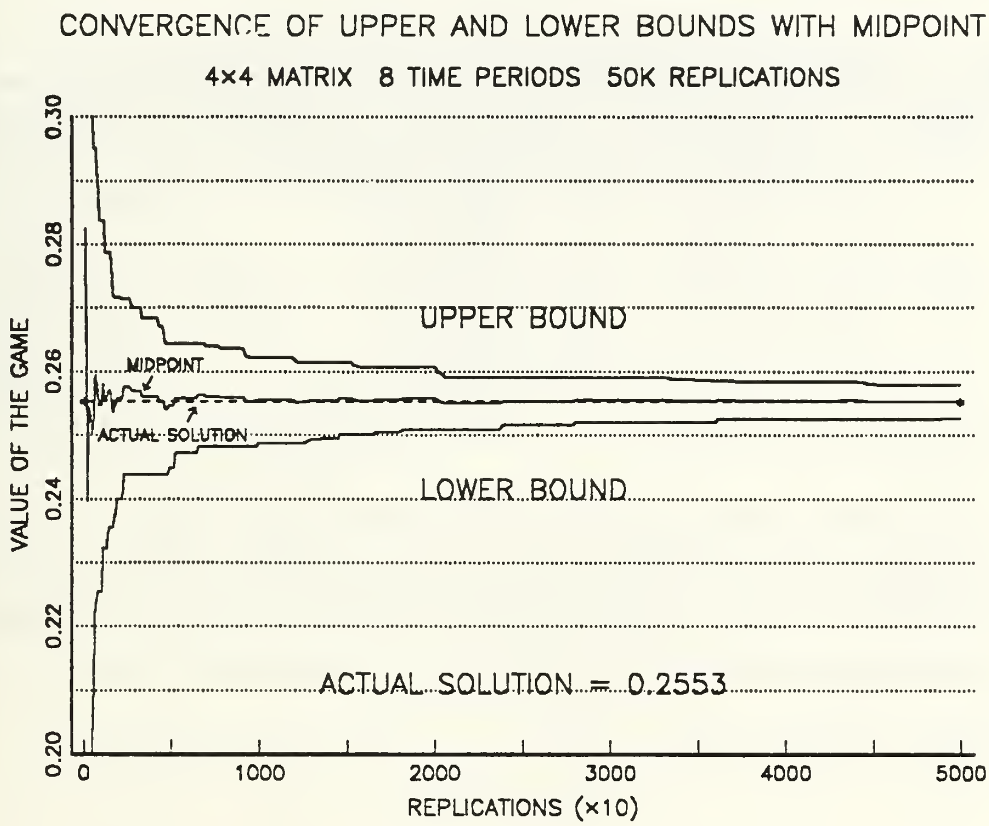

Figure 3. Convergence of Upper and Lower Bounds With Midpoint Solutions

Since the solution to the game lies lies between the bounds and the bounds appear to converge symmetrically, using the midpoint of the bounds as an approximation to the value of the game seemed to be a reasonable approach. This method proved to be very successful as evidenced in Table 1 on page 13. Further investigation revealed that through the use of the midpoint method an accurate approximation could be predicted without requiring a large number of replications. This is supported by a comparison of

the midpoint and actual solution for various replications in Table 2. Additional comparisons are available in Appendix B.

Table 2. MIDPOINT SOLUTIONS FOR MULTIPLE REPLICATIONS

MATRIX SIZE	REPLICATIONS (x1000)	MIDPOINT SOLUTION	ACTUAL SOLUTION	ABSOLUTE DIFFERENCE
4x4	5	0.2547	0.2553	0.0006
	10	0.2556		0.0003
	20	0.2557		0.0004
	30	0.2556		0.0003
	40	0.2555		0.0002
	50	0.2554		0.0001

Normally the fictitious play process is considered complete when the difference between the bounds achieves some specified positive tolerance level. A small tolerance level results in an accurate solution. It was observed that as the game increased in size, more replications were required to obtain the same tolerance level. This is illustrated in Figure 4.

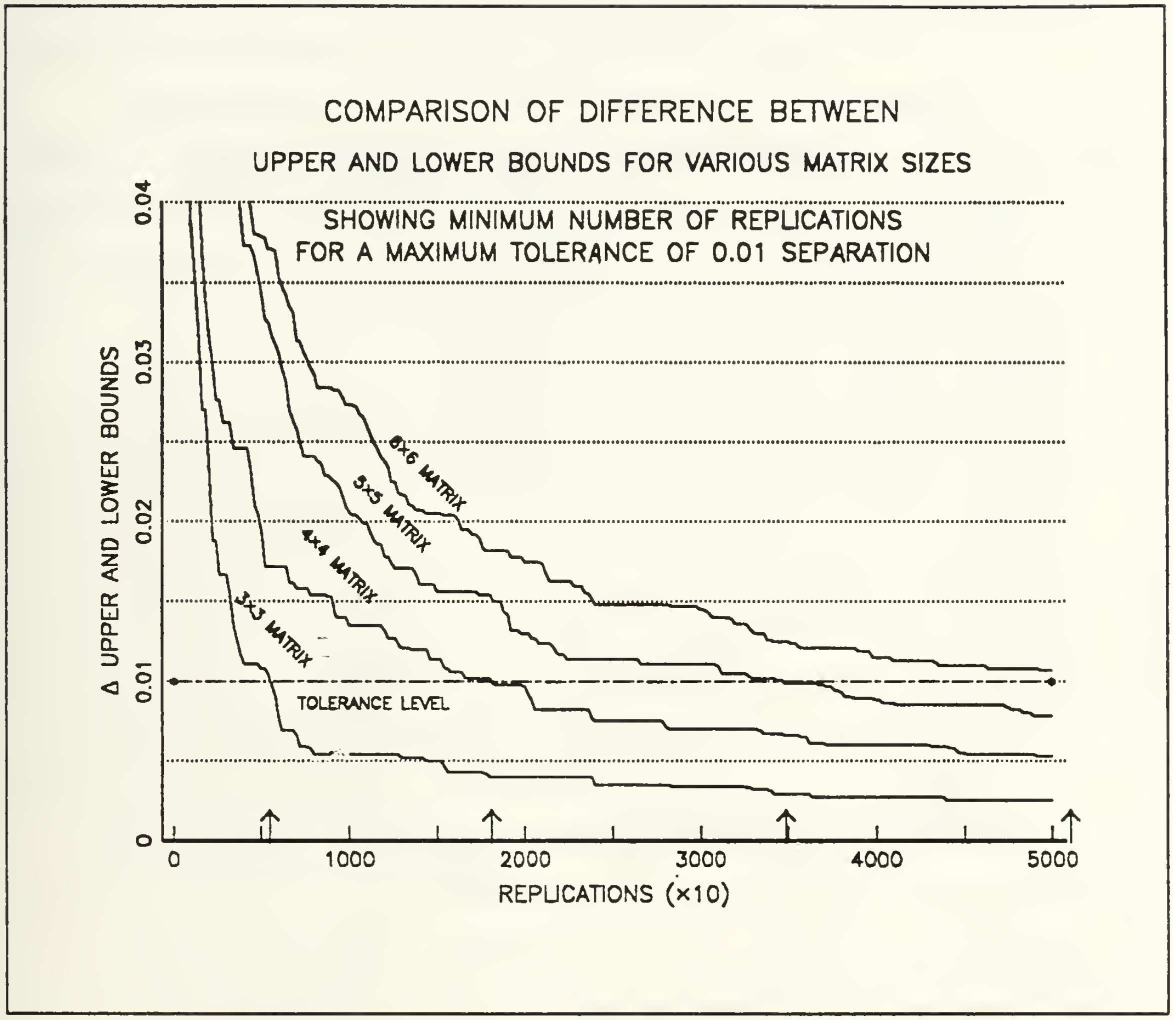

Figure 4. Comparison of Separation Between Bounds for Various Matrix Sizes

Figure 5 on page 18 compares the convergence of the midpoint solution to that of the upper bound for a typical game. The midpoint was in all cases observed to achieve a much more accurate estimate of the value of the game than provided by either of the bounds. For example, 18,000 replications were required in a 4x4 game to bring the upper bound within 0.005 of the actual solution; and only 400 replications were required to bring the midpoint to the same absolute deviation.

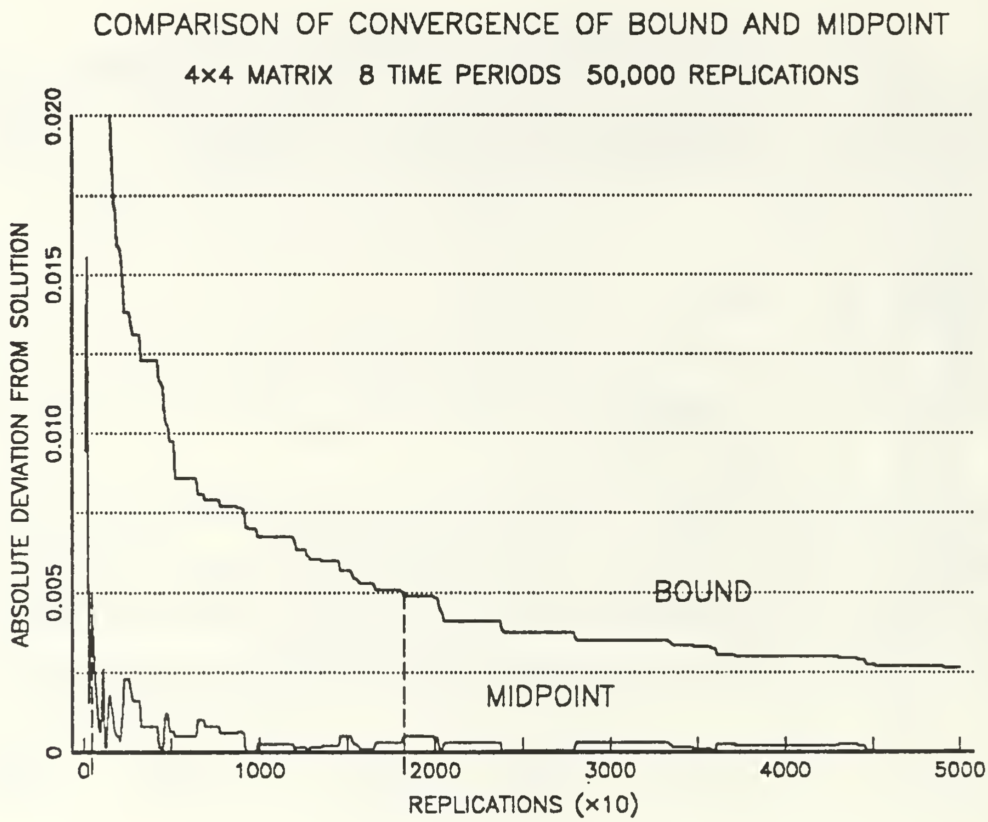

Figure 5. Comparison of Convergence of an Upper Bound and Midpoint

By examining further comparisons in Appendix C, it can be seen that as the game size increases, the number of replications required by the bound to guarantee a preset absolute deviation also increases. The number of replications required to give the same absolution deviation for the midpoint solution appears to be considerably less influenced by increasing game sizes. A comparison between the bound and midpoint replications required to insure a 0.005 absolute deviation for various game sizes is provided in

Table 3. The midpoint method is seen to provide an accurate approximation to the solution very quickly and is not greatly influenced by the game size.

Table 3. REPLICATIONS TO INSURE 0.005 ABSOLUTE DEVIATION FOR BOUND AND MIDPOINT

MATRIX SIZE	REPLICATIONS FOR 0.005 DEVIATION WITH BOUND	REPLICATIONS FOR 0.005 DEVIATION WITH MIDPOINT
3x3	5,500	400
4x4	18,000	400
5x5	34,800	400
6x6	> 50,000	500

It can be concluded from the above observations that for the games examined the bounds converge symmetrically to a solution. The midpoint between the bounds converges much more rapidly to the solution than do the bounds. Additionally, the midpoint method is apparently not greatly hindered by an increase in game size.

C. CONVERGENCE RATE

The convergence rate for the Fictitious Play process is not clearly understood. For the area search game, the convergence rates for various size games were experimentally found to be slower than the hypothesized rate $1/n$. The rate of convergence of the bounds for various games examined and the hypothesized rates are displayed in Figure 6.

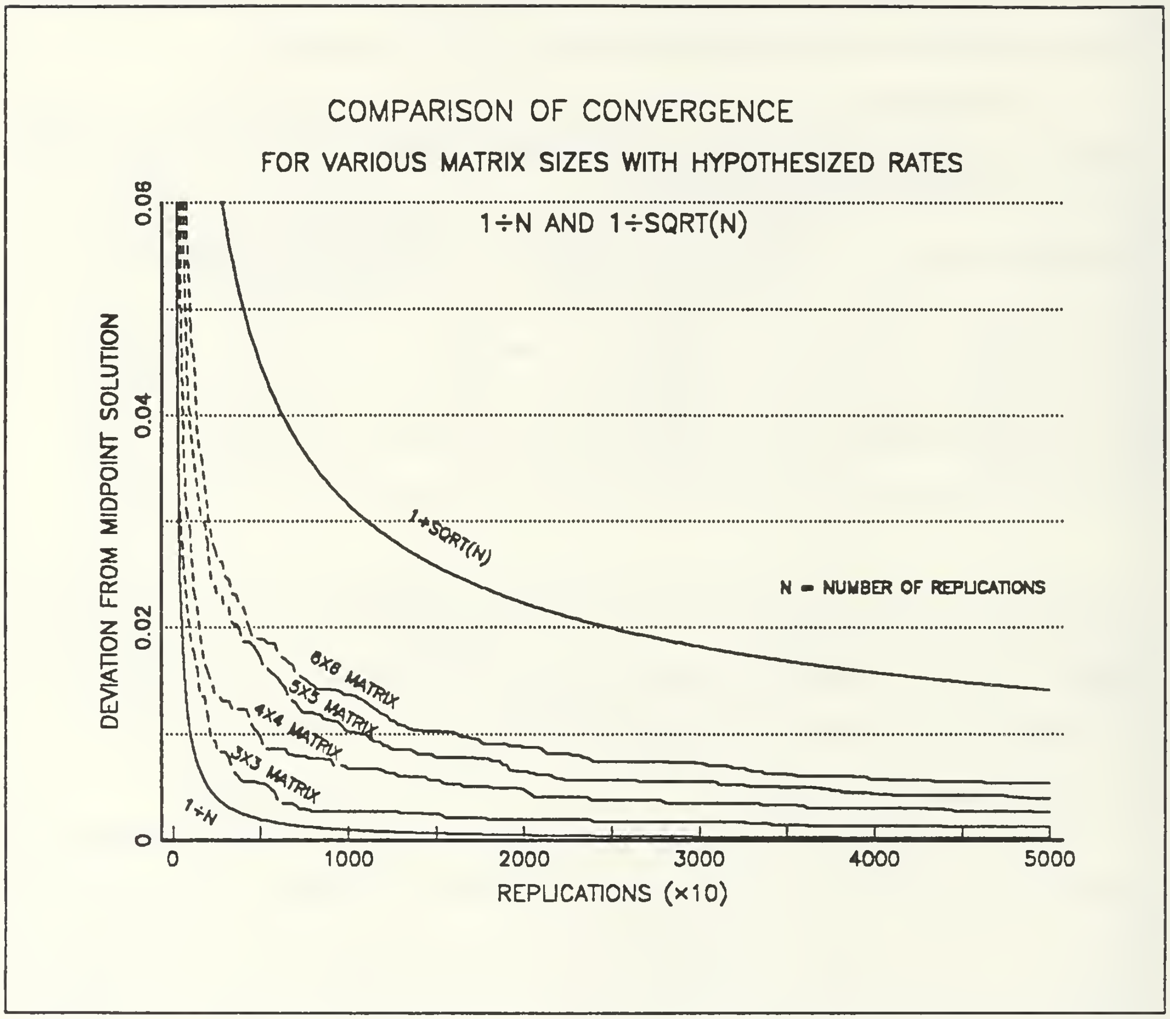

Figure 6. Comparison of Convergence of Upper Bound for Various Game Sizes

It can be clearly seen in Figure 6, that as the game sizes increase the associated time (i.e., number of replications) required to reach a specific deviation also increases. That is, the bounds take longer to converge for a larger games.

Experimentally the data were fitted with the power function ($y = \alpha n^{\beta}$), where n represented the number of replications. This fit was accomplished by noting that for this power function

$$\ln y = \ln \alpha + \beta \ln n$$

which allowed a linear regression of $\ln y$ versus $\ln n$ to be done. Because of the large amount of data, every fiftieth replication was fitted. The exponent (β), which re-

presents the convergence rate, appeared to range from the lower hypothesized value of -1.0 and generally increased as the game size increased. The largest game examined was a 10x10 matrix with only 6,850 replications due to the large amount of computer time required to generate the data. Table 4 provides the values of (α) and (β) for the fitted power functions of various games examined. In all cases, the power function provided an excellent fit to the data. This is supported by the closeness to 1.0 of the multiple correlation coefficient (R SQUARE), displayed in Table 4. A graphical presentation of the fitted data is provided in Appendix C.

Table 4. POWER FUNCTION FIT OF DATA WITH R SQUARE VALUES

MATRIX SIZE	POWER FUNCTION ($y = \alpha n^{\beta}$)		R SQUARE
	α	β	
2x2	0.29366	-0.79710	0.973
3x3	0.29687	-0.65306	0.972
4x4	0.40866	-0.59259	0.993
5x5	0.68224	-0.60602	0.995
6x6	0.65516	-0.56800	0.997
8x8	1.19410	-0.55711	0.993
10x10	0.78094	-0.44950	0.991

It can be concluded for this type of area search game, that the Fictitious Play approach has a convergence rate that is representative of a power function. The observed β values ranged from -0.7971 for the smallest game to -0.4495 for the largest. These data suggest that Brown's hypothesized rate of $1/n$ (i.e., $\beta = -1$) is in general too optimistic. Additionally, Karlin's rate of $1/\sqrt{n}$ (i.e., $\beta = -0.5$) may also be too optimistic for games as large or larger than the 10x10, 20-time period game examined here.

V. CONCLUSIONS

A. SUMMARY OF CONVERGENCE PROPERTIES

The fictitious play approach to the two-person zero-sum area search game was successfully implemented. Convergence properties of the process, for this type of game, were investigated and the following conclusions were reached.

- For the area search games examined, the upper and lower bounds on the value of the game converge symmetrically toward a solution as the number of replications of the game is increased.
- Because of the symmetrical convergence, the midpoint between the bounds provides an accurate approximation to the solution.
- The midpoint solution converges much more quickly to the actual solution than do the bounds and is apparently not greatly influenced by the size of the game.
- The convergence rate of the process is representative of a power function ($y = \alpha n^{\beta}$). Experimentally, the exponent (β) was observed to vary between -0.7971 and -0.4495, and generally increased with the size of the game.
- The convergence of the bounds becomes slower as the size of the game is increased.

By observing the convergence characteristics of the Fictitious Play process, an approach for predicting an accurate approximation of the solution was developed. This approach, the midpoint method, required less replications of the game and provided an accurate approximation of the solution. Because of the increased efficiency and capability to accurately predict a solution, the Fictitious Play process should be considered a possible approach to solving area search games and warrants further investigation.

B. RECOMMENDATIONS FOR FUTURE STUDY

The Fictitious Play process provides a relatively simple approach to solving the area search game. It not only produces an accurate approximation of the solution of the game, but also provides the capability to determine a nearly optimal strategy for each player. The following topic is recommended as a possible area for further investigation and future study.

1. Comparison of Linear Programming and Fictitious Play Approaches

Solving the area search game with the use of computer resources can be approached by several methods. One very promising method is the Linear Programming approach of Washburn. The major advantage of this approach is that it does not require a large amount of CPU time and gives exact answers. However, it can be very

demanding on memory resources, especially as the size of the games increases. In comparison, the fictitious play approach requires minimal memory resources but is hampered by the large amount of CPU time required and gives approximate solutions. A comparison between the Linear Programming and the Fictitious Play approach is recommended for future study, with emphasis on the tradeoffs between the resources of CPU time and memory space.

APPENDIX A. COMPARISONS OF CONVERGENCE OF UPPER AND LOWER BOUNDS ON VALUE OF THE GAME

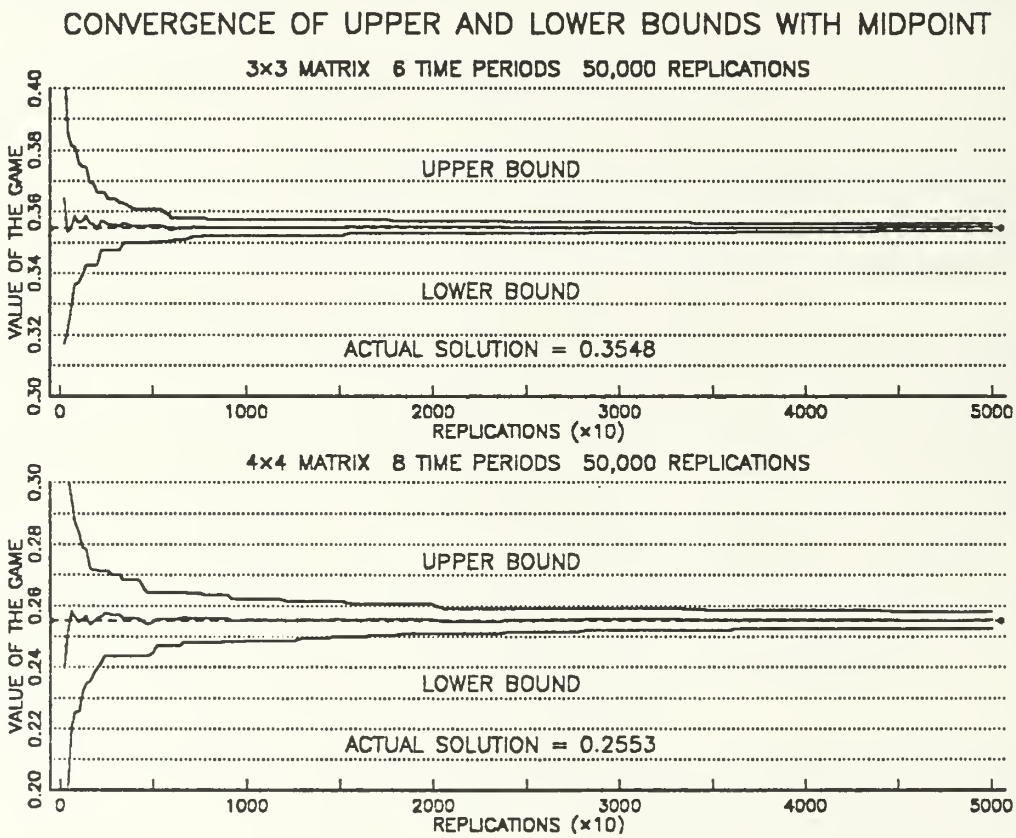

Figure 7. Convergence of Upper and Lower Bounds With Midpoint: 3x3 and 4x4 Matrix

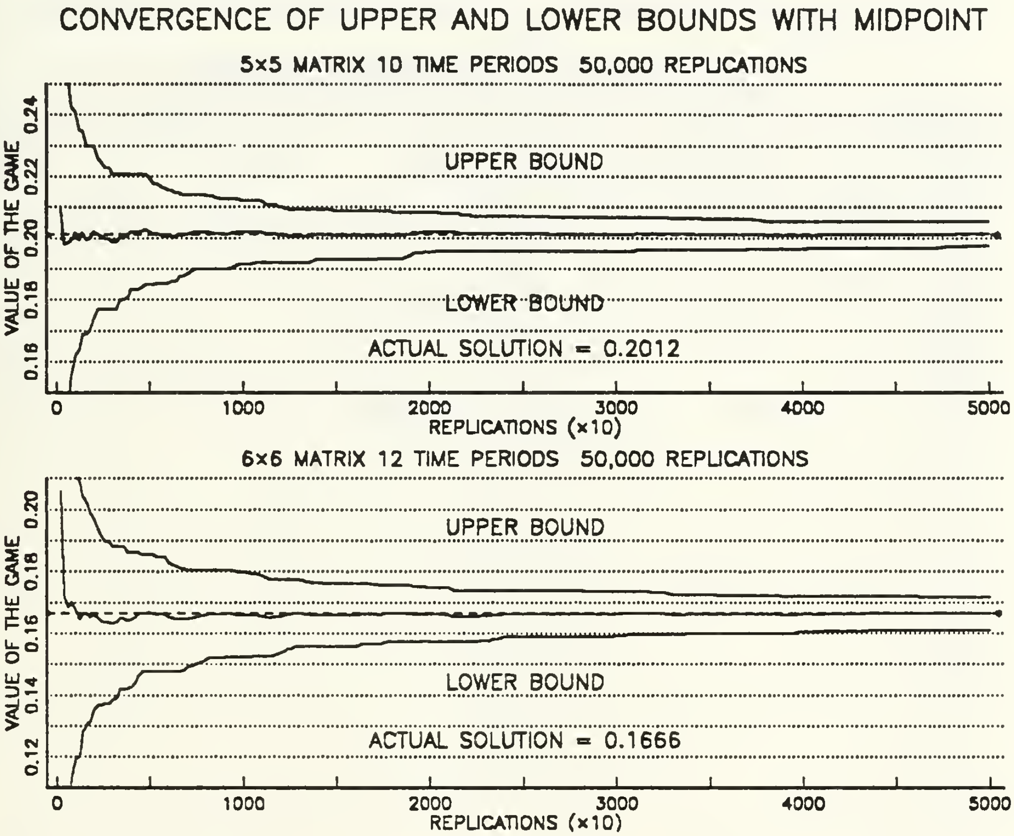

Figure 8. Convergence of Upper and Lower Bounds With Midpoint: 5x5 and 6x6 Matrix

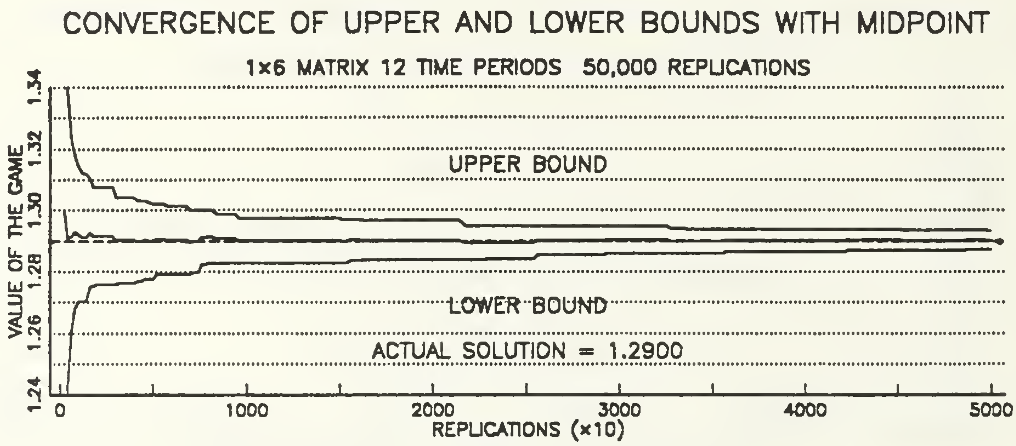

Figure 9. Convergence of Upper and Lower Bounds With Midpoint: 1x6 Matrix

APPENDIX B. COMPARISON OF MIDPOINT AND ACTUAL SOLUTION

Table 5. MIDPOINT SOLUTIONS FOR MULTIPLE REPLICATIONS

MATRIX SIZE	REPLICATIONS (x1000)	MIDPOINT SOLUTION	ACTUAL SOLUTION	ABSOLUTE DIFFERENCE
3x3	5	0.3556	0.3548	0.0008
	10	0.3549		0.0001
	20	0.3551		0.0003
	30	0.3549		0.0001
	40	0.3549		0.0001
	50	0.3550		0.0002
4x4	5	0.2547	0.2553	0.0006
	10	0.2556		0.0003
	20	0.2557		0.0004
	30	0.2556		0.0003
	40	0.2555		0.0002
	50	0.2554		0.0001
5x5	5	0.2024	0.2012	0.0012
	10	0.2020		0.0008
	20	0.2019		0.0007
	30	0.2013		0.0001
	40	0.2010		0.0002
	50	0.2015		0.0003

Table 6. MIDPOINT SOLUTIONS FOR MULTIPLE REPLICATIONS (CONT.)

MATRIX SIZE	REPLICATIONS (x1000)	MIDPOINT SOLUTION	ACTUAL SOLUTION	ABSOLUTE DIFFERENCE
6x6	5	0.1667	0.1666	0.0001
	10	0.1661		0.0005
	20	0.1662		0.0004
	30	0.1664		0.0002
	40	0.1663		0.0003
	50	0.1665		0.0001
1x6	5	1.2899	1.2900	0.0001
	10	1.2902		0.0002
	20	1.2902		0.0002
	30	1.2903		0.0003
	40	1.2901		0.0001
	50	1.2902		0.0002

APPENDIX C. COMPARISON OF CONVERGENCE TO THE ACTUAL SOLUTION OF A BOUND AND MIDPOINT

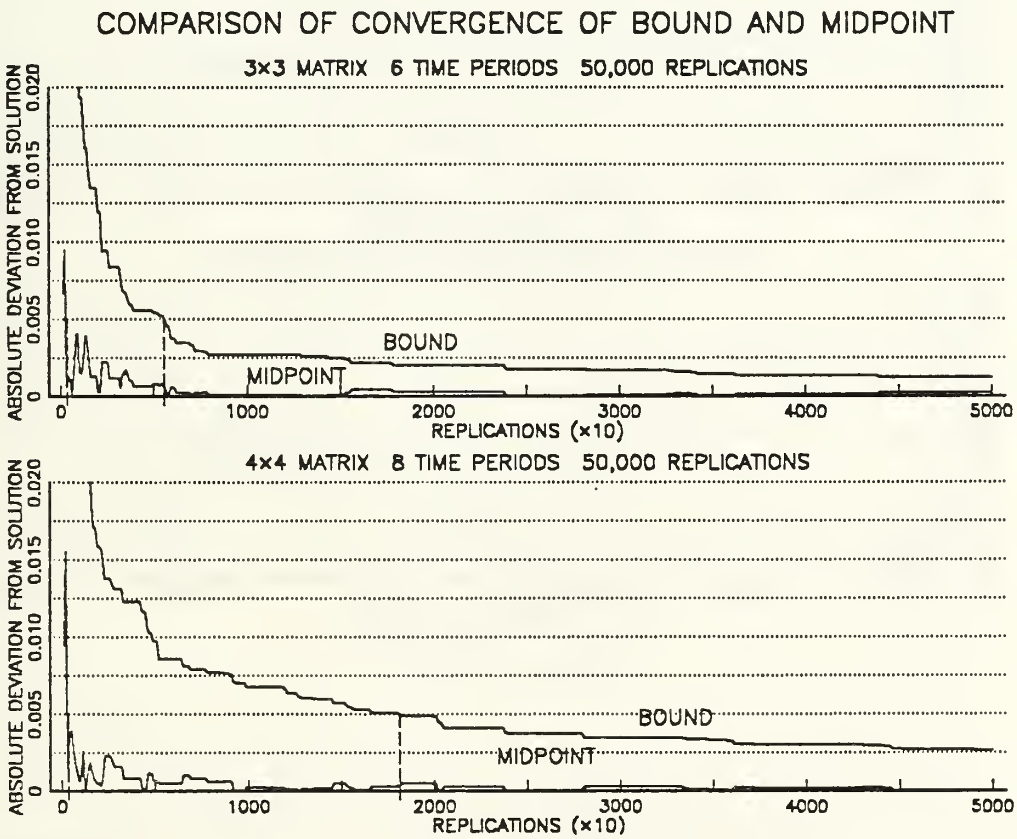

Figure 10. Convergence of a Bound and Midpoint for a 3x3 and 4x4 Matrix Game

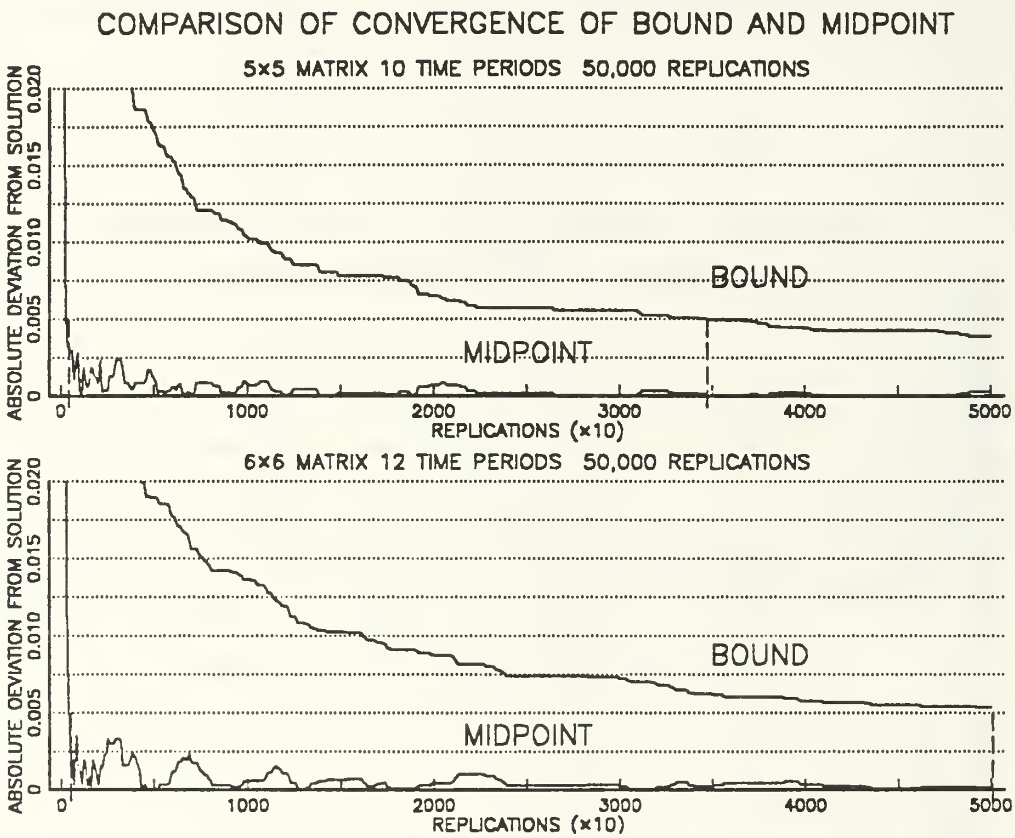

Figure 11. Convergence of a Bound and Midpoint for a 5x5 and 6x6 Matrix Game

APPENDIX D. POWER FUNCTION FITTING OF DATA FOR VARIOUS GAME SIZES

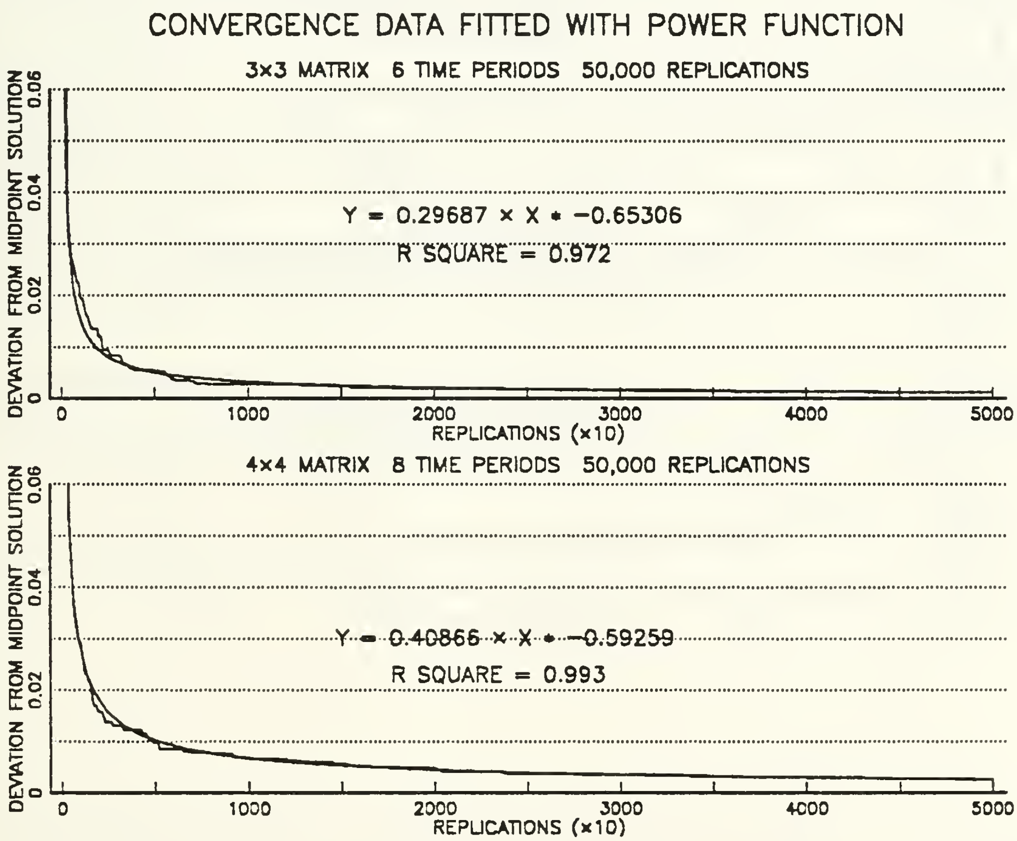

Figure 12. Convergence Data From 3x3 and 4x4 Matrix Fitted With Power Function

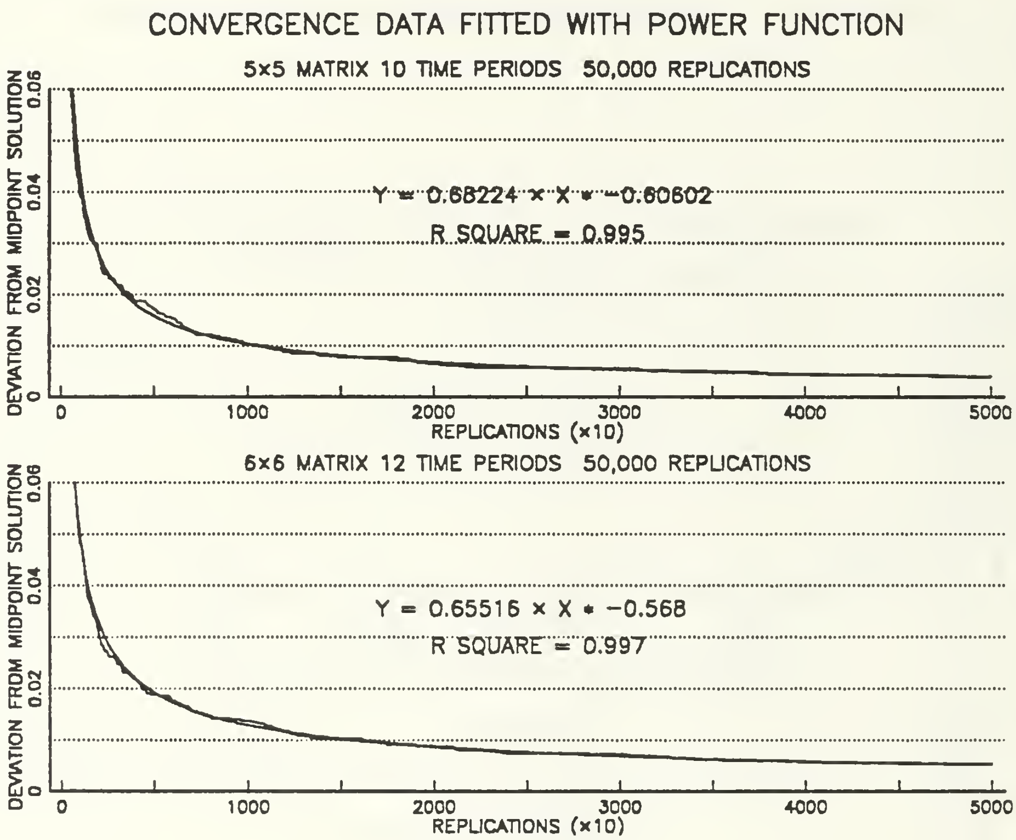

Figure 13. Convergence Data From 5x5 and 6x6 Matrix Fitted With Power Function

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